Ordinary Differential Equations (ODEs)

DEFINITION

An ***ordinary differential equation (ODE)*** is an equation that contains one or several derivatives of an unknown function, which we usually call y(x) (or sometimes called y(t) if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

(1) $y' = \cos x$

(2) $y'' + 9 = e^x$

(3) $y''' - 2y' = 0$

Here, as in calculus, y' denotes dy/dx, $y^2 = d^2y/dx^2$etc. The term *ordinary* distinguishes them from ***partial differential equations*** (PDEs), which involve partial derivatives of an unknown function of *two or more* variables. For instance, a PDE with unknown function u of two variables x and y is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

PDEs have important engineering applications, but they are more complicated than ODEs

An ODE is said to be of order n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (1) is of first order, (2) of second order, and (3) of third order.

First-order ODEs

Linear 1st ODEs

A Differential Equation (D.E.) that can be written in the form:

$$\frac{\partial y}{\partial x} + P(x)\, y = Q(x) \ldots\ldots\ldots \text{Standard Form}$$

Where P and Q are function of x is called a linear first order ordinary differential equation.

Steps for solving……

1) Have the standard form.

2) Get *P(x)* and *Q(x).*

3) Find $\boldsymbol{\rho(x)}$ *where* $====> \boldsymbol{\rho(x)} = e^{\int P(x)dx}$

4) Use the equation $y = \frac{1}{\rho(x)} \int \rho(x)\, Q(x) dx$ to find *y.*

Special Case

If we have:

$$\frac{\partial y}{\partial x} + a\, y = b$$

Where **a**, **b** are **constant** then the solution:-

$$y = C_1\, e^{-ax} + \frac{b}{a}$$

For example:-

$\frac{\partial y}{\partial x} + 5\, y = 10$ ======> $y = C_1\, e^{-5x} + 2$

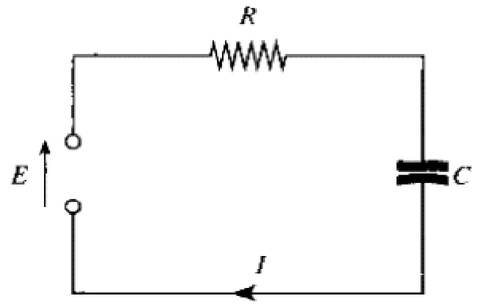
$\frac{\partial y}{\partial x} + y = 1$ ======> $y = C_1\, e^{-x} + 1$

Main Ref. 1. E. Kreyszig. "ADVANCED ENGINEERING MATHEMATICS".
2. L. Finny & B. Thomas, "CALCULAS".

$\frac{\partial y}{\partial x} + 6\,y = 3$ ======> $y = C_1\, e^{-6x} + 0.5$

Example: - for the figure below the key closed at (t=0) write down the instantaneous equation of the current *i* in term of time *t* if u know that i (t=0) = 0?

Solution:-



$$i(t) = \frac{48}{11}\, e^{-110t} + \frac{48}{11} \quad Ampere$$

Reduction to Linear Form
Bernoulli Equation

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the **Bernoulli equation**.

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

Where $n \neq 1, 0$

To transfer such equation let:

$$z = y^{1-n}$$

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$$\frac{dz}{dx} + (1-n)\,P(x)\,z = (1-n)\,Q(x)$$

Exact 1st ODEs

A first-order ODE that can be written in the form:

$$M(x,y)dx + N\,(x,y)dy = 0$$

Test for exact condition where

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ It **is exact**.

If $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ It is **not exact.**

Please, refer to page (A-71) answer in" calculus" ….check the solution in section 16.2 /point (9)….what is your opinion in the given answer

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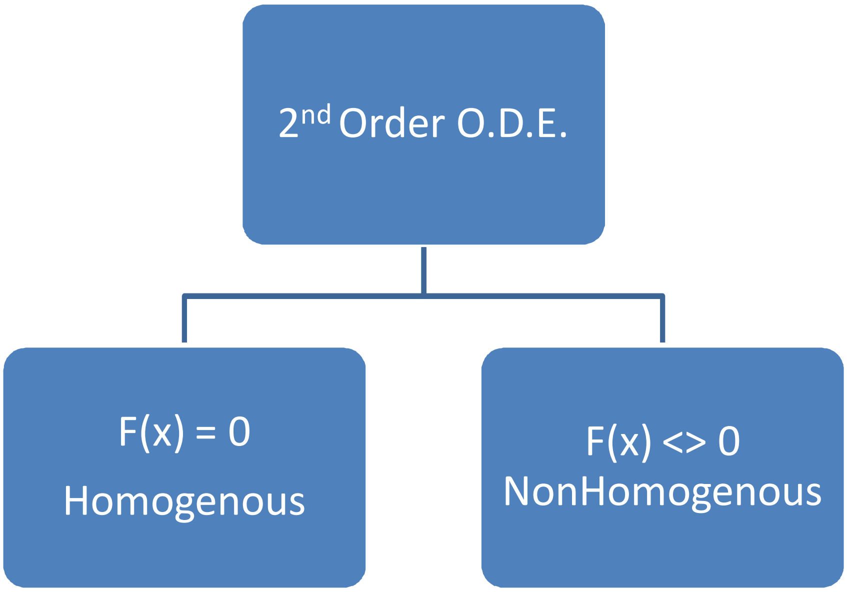
Linear 2nd order ODEs

Many important applications in mechanical and electrical engineering are modeled by linear ordinary differential equations (linear ODEs) of the second order. Their theory is representative of all linear ODEs as is seen when compared to linear ODEs of third and higher order, respectively

However, if we have:-

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}\dots\dots\dots\dots..+a_o(x)y=f(x)$$

$$if\ \ f(x)=0\ \ the\ equation\ called\ \boldsymbol{homogenous}\ otherwise$$
$$its\ called\ \boldsymbol{nonhomogenous}$$



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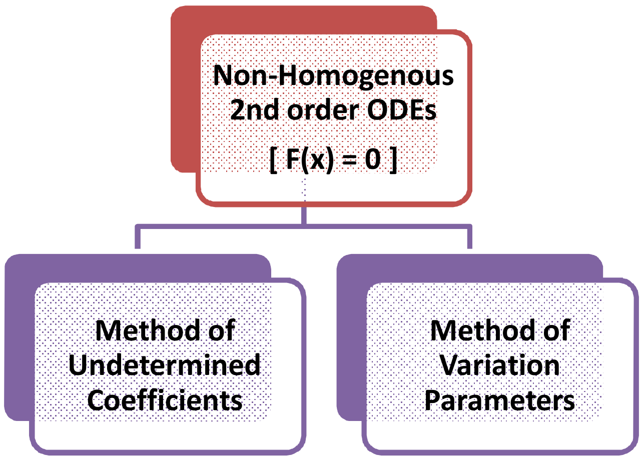
Homogenous 2nd order ODEs F(x) = 0

Case	Roots of (2)	General Solution
I	**Distinct Real** $r_1 \neq r_2$	$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
II	**Real double** $r_1 = r_2 = r$	$y = [C_1 + C_2 x] e^{rx}$
III	**Complex Conjugate** $r = a + jb$	$y = e^{ax}[C_1 \cos bx + C_2 \sin bx]$

Non-Homogenous 2nd order ODEs [F(x) ≠ 0]



Method Undetermined Coefficients

The solution is y_g *(y general) and equal to:-*

$$y_g = y_h + y_p$$

Where:-

$y_h = homogenous\ solution\ of\ the\ given\ O.D.E.$

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y_P = *Particular solution of the given O.D.E.*

Case	C. E.	y_P
e^{rx}	r is not a root in C.E. r is a single root in C.E. r is a Double root in C.E.	Ae^{rx} Axe^{rx} Ax^2e^{rx}
$\sin mx$, $\cos mx$	m is not a root in C.E. m is a single root in C.E. m is a Double root in C.E.	$A\ Cos\ x + B\ Sin\ x$ $A\ x\ Cos\ x + B\ x\ \ Sin\ x$ $A\ x^2\ Cos\ x + B\ x^2\ \ Sin\ x$
$ax^2 + \ bx + c$	0 is not a root in C.E. 0 is a single root in C.E. 0 is a Double root in C.E.	$A\ x^2 + B\ x + C$ $A\ x^3 + B\ x^2 + C\ x$ $A\ x^4 + B\ x^3 + C\ x^2$

Method of Variation Parameters

How to solve using this method??

1) Get the homogenous solution y_h.
2) Put the homogenous solution in the form $y_h = C_1 U_1 + C_2 U_2$.
3) Get D

   Where

$$D = \begin{bmatrix} U_1 & U_2 \\ U_1' & U_2' \end{bmatrix}$$

4) Get V_1 *and* V_2 where

$$V_1 = \int \frac{-U_2 * f(x)}{D} dx$$
$$V_2 = \int \frac{U_1 * f(x)}{D} dx$$

5) Finally write y as

$$y = U_1 V_1 + U_2 V_2$$

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Linear 3rd order ODEs

$$y''' + ay'' + by' + cy = f(x)$$

General Solution is:

$$y_g = y_h + y_p$$

Case (I)

$r_1 \neq r_2 \neq r_3$

$$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x}$$

Case (II)

$r_1 = r_2 = r_3$

$$y_h = [C_1 + C_2 x + C_3 x^2]e^{rx}$$

Case (III)

$r_1 = r_2 \neq r_3$

$$y_h = [C_1 + C_2 x]e^{r_1 x} + C_3 e^{r_3 x}$$

Case (IV)

r_1 & r_2 *are complex number* , $r_3 = real$

$$y = e^{ax}[C_1 \cos bx + C_2 \sin bx] + C_3 e^{r_3 x}$$

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